Polarization characteristics of spontaneous emission and off-axis coherent gain in a free-electron laser

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The polarization characteristics of spontaneous radiation from relativistic electrons moving through helical and planar wiggler fields are evaluated for imperfect beam injection. Maximum coherent gain in free-electron laser systems are seen to occur in optical fields having these polarization characteristics rather than those of the wiggler magnets. Coupling coefficients for an electron beam skewed at an angle to the optical mode are presented.

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I. INTRODUCTION

The polarization characteristics of the amplified radiation in free-electron lasers have been taken to be the same as that of the wiggler magnet for both perfect [1] and imperfect [2] electron beam injections. For perfect injection and helical wigglers, axial electron oscillations do not take place; with planar wigglers, axial oscillations appear at twice the wiggler frequency. For imperfect injection, axial oscillations occur at the wiggler frequency for both the helical and planar wigglers. Such axial oscillations play a significant role in determining the polarization characteristics of spontaneous radiation. However, these characteristics do not appear to have been considered in working out the features of the coherent optical field. Thus, for instance, Colson, Dattoli, and Ciocci [2], working with imperfect injection, have assumed an optical field having the same polarization as that of the wiggler magnet.

We show in this Brief Report that for imperfect electron beam injection in a helical wiggler the axial oscillations lead to an elliptically polarized spontaneous emission, where both the eccentricity and the azimuth of the ellipse are injection dependent; for linearly polarized wigglers the spontaneous emission is also linearly polarized but with an injection dependent azimuth. It is further shown that the optical fields having the same polarization characteristics as those of the spontaneous radiation are optimally amplified.

II. FEL DYNAMICS WITH A CIRCULARLY POLARIZED WIGGLER

A. Spontaneous emission

We consider a wiggler magnetic field $\mathbf{B}_h = B(\cos k_0 z, -\sin k_0 z, 0)$ along the free-electron laser (FEL) axis. The transverse velocity components of an electron (of energy γmc^2) entering the magnetic field with velocity $c\boldsymbol{\beta}$ skewed at an angle to the z axis (i.e., imperfect beam injection [2]) are given by

$$\boldsymbol{\beta}_T = \hat{x}\left[-\frac{K}{\gamma}\cos k_0 z + \theta\cos\psi\right] + \hat{y}\left[\frac{K}{\gamma}\sin k_0 z + \theta\sin\psi\right], \tag{1}$$

where $k_0[=2\pi/\lambda_0=\omega_0/c]$ is the wiggler wave number and $K = eB/mc^2k_0$ is the wiggler parameter. θ and ψ are integration constants, such that Eq. (2.7) of Ref. [2] could be obtained by putting $\psi=0$. $\theta=0$ corresponds to perfect beam injection; under this condition the electrons move along a helix of constant radius K/γ about the FEL axis, the z axis. For imperfect injection, $\theta\neq 0$; the electrons also experience a drift in the transverse (x,y) directions. Thus θ can be looked upon as a measure of deviation from perfect electron beam injection.

The axial electron velocity β_z is obtained by using the relation $\gamma^2 = 1-(\beta_x^2+\beta_y^2+\beta_z^2)$ along with Eq. (1). Thus up to the second order in K/γ ($\gamma \gg 1$)

$$\beta_z = \bar{\beta}_z + \frac{K\theta}{\gamma}\cos(k_0 z+\psi), \quad z = \bar{z} + \frac{K\theta}{\gamma k_0 \beta_0}\sin(k_0 z+\psi), \tag{2}$$

$$\bar{\beta}_z = \left[1-\frac{1}{\gamma_\parallel^2}\right]^{1/2} = \beta_0, \quad \bar{z} = z_0 + \beta_0 ct,$$
$$\gamma_\parallel = \gamma(1+K^2+\gamma^2\theta^2)^{-1/2}, \tag{3}$$

where z_0 is the constant of integration and the overbars denote averages over the wiggler wavelength. The oscillations in z cause emission into higher harmonics, and their amplitude determines χ in the argument of the Bessel functions (see below).

The Fourier transform of the vector potential representing the spontaneous radiation field is obtained, using the standard technique [3], as

$$\mathbf{A}(f\omega_r) = \left[\left[\frac{e^2 f\omega_r^2}{8\pi^2 c}\right]\right]^{1/2} \times \frac{K}{\gamma}[(\hat{x}+i\hat{y})J_a e^{-i(f+1)\psi} + (\hat{x}-i\hat{y})J_b e^{-i(f-1)\psi}]e^{-i(fk_r+fk_0)z_0} \times \frac{\sin[f\omega_r(1-\beta_0)-f\omega_0\beta_0]\dfrac{N\pi}{\omega_0\beta_0}}{[f\omega_r(1-\beta_0)-f\omega_0\beta_0]}, \tag{4}$$

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where $f=1,2,3\ldots$ is the harmonic number, N is the number of undulator periods, J_n is the Bessel function of the first kind, and

$$J_{a,b}=\left[J_{f\pm1}(f\chi)-\left[\frac{\gamma\theta}{K}\right]J_f(f\chi)\right],\quad \chi=\frac{2K\theta\gamma}{1+K^2+\gamma^2\theta^2}. \tag{5}$$

The spontaneous energy radiated at frequency $f\omega_r=f2\gamma^2\omega_0\beta_0/(1+K^2+\gamma^2\theta^2)$ in the forward direction per unit solid angle $(d\Omega)$ per unit frequency interval $(df\omega_r)$ is given by

$$\frac{d^2I}{d\Omega\,df\,\omega_r}=\frac{8(e\gamma N)^2}{c}\sum_{f=1}^{\infty}\left[\frac{fc}{L(1+K^2+\gamma^2\theta^2)}\right]^2\kappa_{fs}^2(\chi)\times\frac{\sin^2[f\omega_r(1-\beta_0)-f\omega_0\beta_0]\frac{N\pi}{\omega_0\beta_0}}{[f\omega_r(1-\beta_0)-f\omega_0\beta_0]^2}, \tag{6}$$

$$\kappa_{fs}(\chi)=K\left[J_{f+1}^2(f\chi)+J_{f-1}^2(f\chi)-\left[\frac{2(1+K^2)}{K^2}\right]J_f^2(f\chi)\right]^{1/2}. \tag{7}$$

Equations (6) and (7) are seen to be independent of the initial phase ψ of the electron's axial oscillation represented by Eq. (2) and have been reported earlier [2]. ψ, however, plays an important role in determining the state of polarization of the spontaneous radiation. Equation (4) leads to the expressions for (i) the ratio $\alpha_s=Q/P$ of the amplitudes of the two transverse radiation field components and (ii) their phase difference $\eta_s(=\theta_2-\theta_1)$ (apart from the factor of $\pi/2$). These are

$$\alpha_s=\pm\frac{F_2}{F_1},\quad \tan\eta_s=-\frac{2J_aJ_b\sin2\psi}{J_a^2-J_b^2},\qquad F_{1,2}=J_a^2+J_b^2\pm2J_aJ_b\cos2\psi. \tag{8}$$

Thus, the spontaneous emission may be written as

$$\mathbf{E}_s=\hat{x}P\cos(f\omega_rt-\theta_1)-\hat{y}Q\sin(f\omega_rt-\theta_2). \tag{9}$$

Viewed along the axis, therefore, the spontaneous emission is elliptically polarized. The azimuth of the ellipse varies with η_s but the eccentricity $e_{fs}=\{1-[(J_a-Jb)/J_a+J_b]^2\}^{1/2}$ remains the same.

If $(\theta=0)$, the axial oscillations are absent, χ vanishes, $(J_a-J_b)/(J_a+J_b)$ becomes unity, e_{fs} goes to zero, and the spontaneous radiation is circularly polarized. Consequently, for perfect injection the spontaneous emission is confined to the fundamental frequency $(f=1)$. Both the emission of harmonic frequencies and ellipticity of the spontaneous radiation are thus related to the presence of axial oscillations $(\theta\neq0)$.

B. Interaction with the optical field

Consider an elliptically polarized radiation field of frequency $f\omega_r(=fck_r, f=1,2,3,\ldots)$ in the FEL interaction region represented by the vector potential

$$\mathbf{A}_r(z,t)=\frac{1}{fk_r}[\hat{x}E_x\sin\xi_1-\hat{y}E_y\cos\xi_2], \tag{10}$$

where $\xi_1=fk_rz-f\omega_rt+\phi$ and $\xi_2=\xi_1+\eta_c$. For $\beta_z\sim1(\gamma\gg1)$ the transverse optical force is negligibly small in comparison to the static magnetic force. Thus the solutions for the electron velocity components and axial displacement are the same as Eqs. (1) and (2). Following the same procedure as in Refs. [1,4] the pendulum equation may be written as

$$\frac{d^2\Phi_f(\tau)}{d\tau^2}=-\frac{eL^2f\omega_0\beta_0}{2mc^3\gamma}F_3\cos[\Phi_f(\tau)+\phi+f\psi+\sigma_3], \tag{11}$$

$$F_3^2=E_x^2F_1^2+E_y^2F_2^2-2E_xE_yF_1F_2\cos(\eta_c-\eta_s), \tag{12}$$

$$\tan\sigma_3=\frac{E_xF_1\sin\sigma_1-E_yF_2\sin(\eta_c+\sigma_2)}{E_xF_1\cos\sigma_1-E_yF_2\cos(\eta_c+\sigma_2)},\qquad \tan(\sigma_2-\sigma_1)=-\frac{2J_aJ_b\sin2\psi}{J_a^2-J_b^2}=\tan\eta_s, \tag{13}$$

where $\tau(=ct/L)$ is the dimensionless time, $\Phi_f(t)=f[(k_r+k_0)\bar{z}-\omega_rt]$ is the dimensionless phase describing the interaction between the electrons and radiation, and $d\Phi_f(\tau)/d\tau=\nu_f(\tau)$ is the slowly evolving dimensionless electron velocity.

To derive the slow evolution of the light wave with its phase averaged over many radiation wavelengths, we solve the wave equation for $\mathbf{A}_r(z,t)$ [Eq. (10)], as described in Ref. [1]. Thus, we get

$$\frac{1}{2}\frac{d}{d\tau}(E_x^2+E_y^2)+i\frac{d\phi}{d\tau}E_x^2+i\frac{d(\phi+\eta_c)}{d\tau}E_y^2=2\pi e\rho LF_3\langle e^{-i(\Phi_f+\phi+f\psi+\sigma_3)}\rangle, \tag{14}$$

where ρ is the electron density. The coupled equations (11) and (14) describe the FEL dynamics. The real part constitutes the laser gain and the imaginary part gives the phase shift.

The gain G_f, defined as the fractional change in radiation amplitude for evolution of the optical wave from $\tau=0$ to $\tau=1$, is given by

$$G_f=\frac{8\pi^2e^2\rho L^2Nf}{\gamma^3mc^2}\kappa_{fc}^2\frac{d}{d\nu_{0f}}\left[\frac{\cos\nu_{0f}-1}{\nu_{0f}^2}\right], \tag{15}$$

where κ_{fc} $(=K/[2(1+\alpha_c^2)]^{1/2}[F_1^2+\alpha_c^2F_2^2-2\alpha_cF_1F_2\cos(\eta_c-\eta_s)]^{1/2})$ is the new coupling factor and is different from the corresponding factor of Ref. [2]. The axial oscillation in electron motion reduces the coupling between the electron and e.m. radiation (cf. Ref. [1]). The reduced optical coupling along the axes of the ellipse are J_a-J_b and J_a+J_b. The azimuth of the ellipse, however, depends upon η_c; $\alpha_c = E_y/E_x$ is the ratio of amplitudes giving rise to elliptically polarized radiation. The coupling factor κ_{fc} varies with η_c and the ratio α_c for the input radiation field. It attains the maximum value $(=\kappa_{fs})$ when $\eta_c = n\pi + \eta_s$ and $\alpha_c = \pm F_2/F_1$ for odd and even values of n, respectively. Under these conditions the eccentricities of the spontaneous and coherent radiations are the same. Thus maximum gain occurs for elliptically rather than circularly polarized [2] radiation.

III. FEL DYNAMICS WITH A LINEARLY POLARIZED WIGGLER

The trajectory of an electron for imperfect beam injection through a linearly polarized wiggler $\mathbf{B}_l = B(0, \sin k_0 z, 0)$ is given by $\boldsymbol{\beta}_T = \hat{x}[(K/\gamma)\cos k_0 z + \theta\cos\psi] + \hat{y}(\theta\sin\psi)$, where $\theta\cos\psi$ and $\theta\sin\psi$ are constants of integration describing the injection angle. The presence of both the transverse velocity components changes the polarization characteristics of the spontaneous emission compared to that in perfect beam injection.

The axial motion is given by

$$z = \bar{z} - \frac{K^2}{8\gamma^2}\frac{\sin 2k_0 z}{\omega_0\beta_0} - \frac{K\theta}{\gamma}\frac{\cos\psi \sin k_0 z}{\omega_0\beta_0}\ ,$$
$$\beta_0 = 1 - \frac{1}{2\gamma^2}\left[1 + \frac{K^2}{2} + \gamma^2\theta^2\right]\ . \tag{16}$$

Equation (16) shows two types [2] of axial oscillations. One is at twice the wiggler frequency (present for perfect injection also). It is responsible for on-axis emission in odd harmonics. The other oscillation is at the wiggler frequency that occurs because of imperfect beam injection. It causes off-axis emission in all the higher harmonics.

The spontaneous energy radiated per unit solid angle per unit frequency interval is

$$\frac{d^2 I}{d\Omega\, df\, \omega_r} = 4\frac{(e\gamma N)^2}{c}\sum_{f=1}^{\infty}\left[\frac{fc}{L\left[1+\frac{K^2}{2}+\gamma^2\theta^2\right]}\right]^2 \kappa_{fs}^2(\chi)\frac{\sin^2[f\omega_r(1-\beta_0) - f\omega_0\beta_0]\frac{N\pi}{\omega_0\beta_0}}{[f\omega_r(1-\beta_0)-f\omega_0\beta_0]^2}\ , \tag{17}$$

$$\kappa_{fs} = K\left[A_{1,f}^2 + \frac{2\gamma\theta\cos\psi}{K}A_{0,f}A_{1,f} + \frac{\gamma^2\theta^2}{K^2}A_{0,f}^2\right]^{1/2}\ ,$$
$$A_{\delta,f} = \sum_{m=-\infty}^{\infty} J_m(\chi_1)[J_{f-2m+\delta}(\chi_2) + J_{f-2m-\delta}(\chi_2)],\quad \delta = 0, 1,$$
$$\chi_1 = -\frac{fK^2}{4\left[1+\frac{K^2}{2}+\gamma^2\theta^2\right]},\quad \chi_2 = -\frac{2fK\gamma\theta\cos\psi}{\left[1+\frac{K^2}{2}+\gamma^2\theta^2\right]}\ . \tag{18}$$

The electric field vector of radiation field resulting from spontaneous emission is

$$\mathbf{E}_s = \hat{x}P\sin[f\omega_r t + f(k_r+k_0)z_0] + \hat{y}Q\sin[f\omega_r t + f(k_r+k_0)z_0]\ . \tag{19}$$

The x and y components of spontaneous radiation field [Eq. (19)] are in the same phase. The ratio α_s of amplitudes of the y and x components is

$$\alpha_s = \frac{Q}{P} = \frac{(\gamma\theta\sin\psi/K)A_{0,f}}{A_{1,f} + (\gamma\theta\cos\psi/K)A_{0,f}}\ . \tag{20}$$

Equation (19) thus shows that the resultant vibration is in the xy plane and is linearly polarized at an angle η_s $(=\pi/2 - \tan^{-1}\alpha_s)$ to the direction of the magnetic field. For perfect beam injection the spontaneous radiation is linearly polarized perpendicular to the direction of magnetic field.

Consider a radiation field linearly polarized at an angle η_c to the direction of the magnetic field. It is represented in terms of vector potential $\mathbf{A}_r(z,t) = -1/fk_r(\hat{x}E_x\cos\xi_1 + \hat{y}E_y\cos\xi_1)$, where $E_y/E_x = \alpha_c$. Proceeding as in Sec. II, we see that the pendulum equation for electron motion is

$$\frac{d^2\Phi_f(\tau)}{d\tau^2} = \frac{f2\omega_0\beta_0 L^2 eE_x K}{2\gamma^2 mc^3} \times\left[A_{1,f} + \frac{\gamma\theta\cos\psi}{K}A_{0,f} + \frac{\alpha_c\gamma\theta\sin\psi}{K}A_{0,f}\right] \times\sin[\Phi_f(\tau)+\phi]\ . \tag{21}$$

Solving Eq. (21) we get the FEL gain as

$$G_f = \frac{4\pi^2 e^2\rho L^2 Nf}{\gamma^3 mc^2}\kappa_{fc}^2\frac{d}{d\nu_{0f}}\left[\frac{\cos\nu_{0f} - 1}{\nu_{0f}^2}\right] \tag{22}$$

at the frequency $f\omega_r \cong f2\gamma^2\omega_0\beta_0/(1+K^2/2+\gamma^2\theta^2)$. Here

$$\kappa_{fc}=\frac{1}{\sqrt{(1+\alpha_c^2)}}\left[A_{1,f}+\frac{\gamma\theta\cos\psi}{K}A_{0,f}+\frac{\alpha_c\gamma\theta\sin\psi}{k}A_{0,f}\right]^{1/2}$$

is the new factor coupling the electron motion and coherent radiation. As in the case of the helical wiggler, the coupling factor again becomes maximum ($=\kappa_{fs}$) for $\alpha_c=\alpha_s$ and $\eta_c=\eta_s$. Thus we find that the amplification of the input radiation is maximum when it is linearly polarized in the same direction as that of the spontaneous radiation.

IV. DISCUSSION

The FEL gain with the helical wiggler in Ref. [2] has been evaluated for a particular case with a circularly polarized radiation field, i.e., for $E_x=E_y$ and $\eta_c=0$. In this situation the gain is

$$G_f=\frac{8\pi^2e^2L^2N\rho f}{\gamma^3mc^2}\left[J_{f-1}(f\chi)-\frac{\gamma\theta}{K}J_f(f\chi)\right]\times\frac{d}{d\nu_{0f}}\left[\frac{\cos\nu_{0f}-1}{\nu_{0f}^2}\right]. \quad (23)$$

The second term of Eq. (23) involving θ (corresponding to imperfect injection) has been neglected in the gain expression derived in Ref. [2]. This term is not negligible. Its omission leads to an error of more than 10% for the fundamental frequency for the values considered in Ref. [2], i.e., $K=1$, $\gamma=10^2$, and $\theta=10^{-3}$. For the fifth harmonic the error is nearly 13%. If θ is increased fivefold the errors shoot up to 68% and 71%, respectively. For these parameters the gains of elliptically polarized radiation [Eq. (15) for $\kappa_{fc}=\kappa_{fs}$] for the fundamental and fifth harmonic are, respectively, 10% and 11% larger than those [Eq. (23)] of the circularly polarized radiation at the corresponding frequencies. For $\theta=0.005$ these enhancements are 33% and 36%, respectively.

For the planar wiggler the terms involving θ on the right-hand side of Eq. (21) have again been neglected in Ref. [2], as the corresponding terms in the case of helical wigglers. The present work indicates that these terms are likely to lead to appreciable changes in the FEL gain, as in the case of helical wigglers as discussed above.

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